

# Fixed order LPV controller design for LPV models in input-output form

V. Cerone, D. Piga, D. Regruto, R. Tóth

**Abstract**—In this work, a new synthesis approach is proposed to design fixed-order  $H_\infty$  controllers for linear parameter-varying (LPV) systems described by input-output (I/O) models with polynomial dependence on the scheduling variables. First, by exploiting a suitable technique for polytopic outer approximation of semi-algebraic sets, the closed loop system is equivalently rewritten as an LPV I/O model depending affinely on an augmented scheduling parameter vector constrained inside a polytope. Then, the problem is reformulated in terms of bilinear matrix inequalities (BMI) and solved by means of a suitable semidefinite relaxation technique.

## I. INTRODUCTION

The *linear parameter-varying* (LPV) modeling paradigm is nowadays considered as a powerful alternative to derive mathematical descriptions of nonlinear/time-varying phenomena. In the last decades, significant research efforts were devoted to modeling, identification and control of this class of models (see, e.g., the books [1], [2] and the survey paper [3] for a thorough review of the literature on the subject).

Although most of the approaches available in the literature for LPV control design require models in state-space (SS) form, where the matrices describing the state-space equations depend statically and often linearly on the scheduling signals, a large part of the literature on identification of LPV systems refers to input-output (I/O) model structures (see, e.g., [1] for an up-to-date overview). This is due to the fact that the I/O framework allows to extend the widely studied LTI prediction-error approaches to the case of LPV systems avoiding the curse of dimensionality present in the identification of state-space forms. Unfortunately, exact minimal SS realizations of identified I/O LPV models require system matrices which depend not only on the current value of the scheduling signal, but also on a number of its past samples (dynamic dependence), as pointed out in [4]. The problem of dynamic dependence is usually bypassed by applying the linear time invariant (LTI) realization theory to the identified I/O LPV model (see, e.g., [5]), which unfortunately leads to SS realizations which are not exact, and the introduced realization error might be the source of significant performance degradation of the closed-loop

system. In order to overcome this problem, a couple of contribution have recently appeared addressing the problem of controller design for LPV systems described in input-output form ([6], [7]). Such approaches allow the designer to overcome some other limitations of state-space approaches like: (i) the difficulty in the a-priori imposition of a fixed structure to the controller, (ii) the explosion of the matrix dimension involved in the synthesis in case of large scale systems. In [6], the authors derive sufficient conditions for guaranteeing quadratic stability and a prescribed level of  $\mathcal{H}_\infty$  performance of the closed-loop system, under the quite general assumption that the designed feedback control system depends polynomially on the scheduling variable. The derived conditions are written in terms of a stable polynomial, called the central polynomial, which must be selected *a-priori* by the user. Once the central polynomial is chosen, the controller is designed by solving a *sum-of-squares matrix polynomial optimization*. Unfortunately, as clearly stated in [6], the a priori selection of the central polynomial can significantly affect the performance of the obtained closed-loop system. Besides, the computational complexity of the sum-of-squares matrix polynomial optimization could be prohibitive for problems with medium/large size. In order to overcome these limitations, a new approach is proposed in [7] where the selection of the central polynomial is optimized with respect to the achieved closed-loop performance. The problem is then formulated in terms of BMI and solved by exploiting a gradient-based algorithm which is not guaranteed, in general, to converge to the global optimum of the nonconvex BMI problem. Furthermore, although the approach proposed in [7] is less computational demanding than the one considered in [6], its applicability is restricted to the case where (i) the closed loop system depends affinely on the scheduling parameter and (ii) the scheduling parameter ranges within a given polytope.

In this work, we propose an alternative approach to solve the same problem considered in [6] which, unlike the procedure presented in that paper, does not require a-priori selection of the central polynomial. Besides, the proposed approach overcomes the limitations of [7] mentioned above. First, by relying on a recent result by some of the authors in [8] about polytopic outer approximation of semialgebraic sets, the closed loop system with polynomial dependence on the scheduling signal is rewritten as a new input-output model depending affinely on an augmented scheduling parameter vector ranging inside a polytope. Then, thanks to a suitable vertex result, the problem is reformulated in terms of a finite dimensional bilinear matrix inequalities

V. Cerone and D. Regruto are with the Dipartimento di Automatica e Informatica, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy; e-mail: [vito.cerone@polito.it](mailto:vito.cerone@polito.it), [diego.regruto@polito.it](mailto:diego.regruto@polito.it)

D. Piga is with the Delft Center for Systems and Control, Delft University of Technology, Mekelweg 2, Delft, 2628 CD, The Netherlands; email: [D.Piga@tudelft.nl](mailto:D.Piga@tudelft.nl)

R. Tóth is with the Control Systems Group, Department of Electrical Engineering, Eindhoven University of Technology, Potentiaal 4.29, 5600 MB Eindhoven, The Netherlands; email: [R.Toth@tue.nl](mailto:R.Toth@tue.nl)

(BMI) optimization problem and solved by means of a semidefinite relaxation approach. The main novelties of the proposed approach with respect to the results of paper [7] are: (i) we consider a general polynomial dependence on the scheduling signal and (ii) the semidefinite relaxation approach is guaranteed to converge to the global optima of the formulated nonconvex BMI problem. Furthermore, stability of the central polynomial is guaranteed by adding suitable polynomial constraints to the BMI problem.

The paper is organized as follows. Section II is devoted to the problem formulation. Previous results on quadratic stability and  $\mathcal{H}_\infty$  control of LPV input-output models are briefly reviewed in Section III. A new procedure for the solution to the considered problem is then proposed in Section IV. Effectiveness of the proposed approach is shown by means of a simulation example in Section V. Concluding remarks are presented at the end of the paper.

## II. PROBLEM FORMULATION

Consider the feedback control configuration depicted in Fig. 1. The plant to be controlled is assumed to be described by a SISO discrete-time, LPV input-output model  $\mathcal{G}(q^{-1}, \lambda)$  defined by the following linear difference equation:

$$\mathcal{A}(q^{-1}, \lambda(t))y(t) = \mathcal{B}(q^{-1}, \lambda(t))u(t), \quad (1)$$

where  $q^{-1}$  is the backward time-shift operator, *i.e.*,  $q^{-1}y(t) = y(t-1)$ ,  $u(t) : \mathbb{Z} \rightarrow \mathbb{R}$  is the command input signal,  $y(t) : \mathbb{Z} \rightarrow \mathbb{R}$  is the output signal, and  $\lambda(t) : \mathbb{Z} \rightarrow \mathbb{R}^\mu$  is the scheduling variable which, according to the LPV modeling and control literature (see, *e.g.*, [3]) is assumed to be measurable. The scheduling variable  $\lambda(t)$  is assumed to belong to a generic semialgebraic set  $\Lambda \subset \mathbb{R}^\mu$  described by polynomial inequalities. In order to simplify notation, in the rest of the paper the following shorthand notation will be adopted for a generic signal:  $\pi_t \triangleq \pi(t)$ . Furthermore,  $\mathcal{A}(\cdot)$  and  $\mathcal{B}(\cdot)$  are polynomials in the backward shift operator  $q^{-1}$  described as

$$\mathcal{A}(q^{-1}, \lambda_t) = 1 + a_1(\lambda_t)q^{-1} + \dots + a_{n_a}(\lambda_t)q^{-n_a}, \quad (2a)$$

$$\mathcal{B}(q^{-1}, \lambda_t) = b_0(\lambda_t) + b_1(\lambda_t)q^{-1} + \dots + b_{n_b}(\lambda_t)q^{-n_b}, \quad (2b)$$

where  $n_a, n_b \geq 0$  and the coefficients  $a_i$  and  $b_j$  are assumed to be polynomial static functions of  $\lambda_t = [\lambda_{t_1} \ \lambda_{t_2} \ \dots \ \lambda_{t_\mu}]$ . Note that this structure is general enough to represent/approximate a large variety of nonlinear systems in an LPV form. In this paper, we address the problem of designing a fixed structure LPV controller  $\mathcal{K}(q^{-1}, \lambda)$  to guarantee closed-loop stability in terms of the control configuration given in Fig. 1 as well as a prescribed level of performance in terms of the  $L_2$ -induced gain of a chosen

closed-loop polynomial. Regarding the closed-loop stability, we refer in this work to the notion of *quadratic stability* of a LPV system in input-output form introduced by the following definition (see also [6] and [1]).

*Definition 1: (Quadratic stability of LPV systems described by an I/O representation)*

Define the latent variable

$$x(t) = [y(t) \ y(t-1) \ \dots \ y(t-n_a+1)]^\top \quad (3)$$

which is considered as a state variable for the autonomous part of the system depicted in Fig. 1. The closed-loop LPV system in Fig. 1 is said to be quadratically stable if and only if there exists a symmetric matrix  $P = P^\top \succ 0$ ,  $P \in \mathbb{R}^{n_a \times n_a}$ , such that the quadratic function

$$V(t) = x(t)^\top P x(t) \quad (4)$$

is a Lyapunov function which guarantees Lyapunov stability of the system, for all possible trajectories of the scheduling variables inside the compact set  $\Lambda \subset \mathbb{R}^\mu$ . ■

The performance requirement considered in this paper is given in terms of a  $L_2$ -induced gain condition:

$$\|\mathcal{T}_{wz}(q^{-1}, \lambda)\|_{i,2} \leq \gamma, \quad \forall \lambda \in \Lambda, \quad (5)$$

where  $\gamma \geq 0$  is a user-defined constant,  $\|\cdot\|_{i,2}$  is the  $L_2$ -induced gain of a nonlinear system (see, *e.g.*, [9]) and  $\mathcal{T}_{wz}(q^{-1}, \lambda)$  denotes a generic closed-loop systems with the following input-output description:

$$\mathcal{V}(q^{-1}, \lambda(t))w(t) = \mathcal{D}(q^{-1}, \lambda(t))z(t), \quad (6)$$

where

$$\begin{aligned} \mathcal{D}(q^{-1}, \lambda_t) &= 1 + d_1(\lambda_t)q^{-1} + \dots + d_{n_d}(\lambda_t)q^{-n_d}, \\ \mathcal{V}(q^{-1}, \lambda_t) &= v_0(\lambda_t) + v_1(\lambda_t)q^{-1} + \dots + v_{n_v}(\lambda_t)q^{-n_v}. \end{aligned} \quad (7)$$

The function  $\mathcal{T}_{wz}(q^{-1}, \lambda)$  can be, *e.g.*, the sensitivity function or the complementary sensitivity function, possibly weighted by a rational frequency-dependent function  $W_p(z)$ , as commonly done in the framework of  $\mathcal{H}_\infty$  control (see, *e.g.*, [10]). In fact, by using the symbol  $\|\cdot\|_2$  for denoting the  $\ell_2$  norm of a signal, condition (5) can be equivalently written as

$$\|z(t)\|_2 \leq \gamma \|w(t)\|_2, \quad (8)$$

and, as is well known, the considered  $L_2$ -induced gain can be considered as a generalization of the notion of the  $\mathcal{H}_\infty$  norm to the class of LPV systems. The LPV controller  $\mathcal{K}(q^{-1}, \lambda)$ , which is designed directly in the input-output form, is described by the following difference equation:

$$\mathcal{M}(q^{-1}, \lambda, t)u(t) = \mathcal{L}(q^{-1}, \lambda, t)e(t), \quad (9)$$

with

$$\begin{aligned} \mathcal{M}(q^{-1}, \lambda, t) &= 1 + m_1(\lambda, t)q^{-1} + \dots + m_{n_m}(\lambda, t)q^{-n_m}, \\ \mathcal{L}(q^{-1}, \lambda, t) &= l_0(\lambda, t) + l_1(\lambda, t)q^{-1} + \dots + l_{n_\ell}(\lambda, t)q^{-n_\ell}, \end{aligned}$$

where  $n_m, n_\ell \geq 0$  and the functions  $m_i$  and  $l_j$  are assumed to be polynomial in the scheduling variable  $\lambda$ . The coefficients of the polynomial functions  $m_i$  and  $l_j$  are the controller

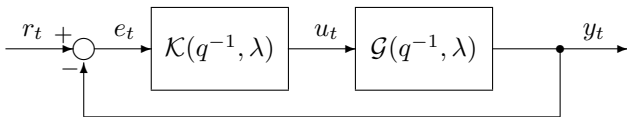


Fig. 1. The considered closed-loop LPV system

parameters to be designed. Note that the controller given in (9) can dynamically depend on the scheduling parameter  $\lambda$ . Denoting the vector of such parameters with the symbol  $\theta$ , we can define the set  $\mathcal{D}_\theta^{qs}$  of all the values of  $\theta$  which guarantee quadratic stability and make the closed loop system satisfying condition (5).

### III. QUADRATIC STABILITY AND $L_2$ GAIN OF LPV SYSTEMS IN INPUT-OUTPUT FORM

The aim of this section is to briefly summarize some results on quadratic stability and  $L_2$ -induced gain of LPV systems given in I/O representation, originally reported in [6], that will be used in the rest of the paper.

Let us define the following quantities

$$\bar{d} = [d_{n_d}(\lambda) \quad \dots \quad d_2(\lambda) \quad d_1(\lambda) \quad 1], \quad (11)$$

$$\bar{v} = [v_{n_v}(\lambda) \quad \dots \quad v_2(\lambda) \quad v_1(\lambda) \quad v_0(\lambda)], \quad (12)$$

$$\Pi_1 = \begin{bmatrix} 0 & 1 & \ddots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \ddots & 1 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 1 & \ddots & 0 & 0 \\ & \ddots & & \vdots \\ 0 & \ddots & 1 & 0 \end{bmatrix},$$

where  $\Pi_1, \Pi_2 \in \mathbb{R}^{n_d \times n_d+1}$ . The following theorem provides a sufficient condition for guaranteeing that the closed-loop LPV system in Fig. 1 is quadratically stable and satisfies the  $L_2$ -induced gain performance condition given by (5).

**Theorem 1:**  $L_2$ -induced gain performance [6].

Given a polynomial  $\mathcal{C}(q^{-1})$  of order  $n_d$  in  $q^{-1}$  and with constant coefficients

$$\mathcal{C}(q^{-1}) = 1 + c_1 q^{-1} + \dots + c_{n_d} q^{-n_d}, \quad (13)$$

with roots inside the unite circle. The LPV system in Fig. 1 is quadratically stable in  $\Lambda$  and satisfies the performance requirements in (5), if and only if there exists a symmetric matrix  $P \in \mathbb{R}^{n_d \times n_d}$  and a scalar  $\beta$  such that the following condition is satisfied for all  $\lambda \in \Lambda$ :

$$\mathcal{J}(\theta, \lambda) \succeq 0, \quad (14)$$

where

$$\mathcal{J}(\theta, \lambda) = \begin{bmatrix} \bar{c}^\top \bar{d}(\lambda) + \bar{d}^\top(\lambda) \bar{c} - F(P) - \beta \bar{c}^\top \bar{c} & \bar{v}^\top(\lambda) \\ \bar{v}(\lambda) & \beta \gamma^2 \end{bmatrix}$$

$$\bar{c} = [c_{n_d} \quad \dots \quad c_2 \quad c_1 \quad 1],$$

and  $F(P) = \Pi_1^\top P \Pi_1 - \Pi_2^\top P \Pi_2$ . ■

The stable polynomial  $\mathcal{C}(q^{-1})$  given in Theorem 1 is referred to as the *central polynomial*.

### IV. LPV CONTROLLER DESIGN PROCEDURE

It is important to highlight that, for given constant polynomial coefficients  $\bar{c}$ , condition (14) is a linear matrix inequality (LMI) in  $\theta$ , in the entries of the matrix  $P$  and in the variable  $\beta$ , and polynomially parameterized in the scheduling parameter  $\lambda$ . Therefore, (14) leads to a semi-infinite LMI constraint. In [6] condition (14) is used to design robust controllers under the quite general assumption that  $\Lambda$  is a semialgebraic set. After selecting a stable polynomial

$\mathcal{C}(q^{-1})$ , the problem of computing the controller parameters satisfying (14) for all  $\lambda$  belonging to  $\Lambda$  is solved in terms of sum-of-squares matrix polynomial optimization (see [6], [11] and references therein). Unfortunately, as discussed in [12], deriving a general formula for the selection of the central polynomial is a hard task, therefore, the user is often forced to look for an appropriate central polynomial by means of trial-and-error iterations. In order to overcome this limitation of the procedure, we propose to design the controller by solving the following optimization problem

$$\hat{\theta} = \arg \min_{\theta, P, \bar{c}, \beta} \beta \quad (15a)$$

s.t.

$$\begin{bmatrix} \bar{c}^\top \bar{d}(\lambda) + \bar{d}^\top(\lambda) \bar{c} - F(P) - \beta \bar{c}^\top \bar{c} & \bar{v}^\top(\lambda) \\ \bar{v}(\lambda) & \beta \gamma^2 \end{bmatrix} \succeq 0 \quad \forall \lambda \in \Lambda \quad (15b)$$

$$\bar{c} \in \mathcal{S}_c \quad (15c)$$

where  $\mathcal{S}_c$  is the set of all stable polynomial of order  $n_d$ , and the coefficients of the central polynomial  $\mathcal{C}(q^{-1}) \in \mathcal{S}_c$  enter the problem as optimization variables. As pointed out in [13], the region  $\mathcal{S}_c$  can be described by means of a set of scalar polynomial inequalities in the coefficients of  $\mathcal{C}(q^{-1})$ . In particular, the following result holds.

**Result 1: Algebraic geometry characterization of stable polynomials [13].**

$\mathcal{S}_c$  is a semialgebraic set defined by the following polynomial inequalities in the coefficients  $c_1, \dots, c_{n_d}$ :

$$\mathcal{C}(1) > 0, \quad (-1)^{n_d} \mathcal{C}(-1) > 0, \quad |c_{n_d}| < 1, \quad (16)$$

$$r_{n_d-1}^2 < r_0^2, \quad s_{n_d-2}^2 < s_0^2, \quad \dots, \quad q_2^2 < q_0^2, \quad (17)$$

where  $r_0, s_0, \dots, q_0, \dots, r_{n_d-1}, s_{n_d-2}, \dots, q_2, q_1$  are elements of the Jury's array reported in Table I. ■

Since the constraint (15b) is a nonconvex BMI in  $\bar{c}$  and  $\theta$ , polynomially parameterized in  $\lambda$ , problem (15) is a semi-infinite nonconvex *polynomial matrix inequalities* (PMI) problem which is difficult to be solved in general. In order to overcome this difficulty, let us consider the following alternative description of the closed-loop polynomial  $\mathcal{T}_{wz}(q^{-1}, \tilde{\lambda})$ :

$$\tilde{\mathcal{V}}(q^{-1}, \tilde{\lambda}(t))w(t) = \tilde{\mathcal{D}}(q^{-1}, \tilde{\lambda}(t))z(t), \quad (18)$$

where  $\tilde{\mathcal{V}}$  and  $\tilde{\mathcal{D}}$  are assumed to depend affinely on a new augmented scheduling variable  $\tilde{\lambda} \in \tilde{\Lambda} \subset \mathbb{R}^\nu$ , and  $\tilde{\Lambda}$  is a suitably defined semialgebraic set which accounts for the fact that the function  $\mathcal{T}_{wz}$ , which is polynomial in  $\lambda \in \Lambda$ , is now written as an affine function of  $\lambda \in \tilde{\Lambda}$ . It is worth noting

TABLE I  
JURY'S ARRAY.

$c_{n_d}$	$c_{n_d-1}$	$c_{n_d-2}$	$\dots$	$c_2$	$c_1$	1
1	$c_1$	$c_2$	$\dots$	$c_{n_d-2}$	$c_{n_d-1}$	$c_{n_d}$
$r_{n_d-1}$	$r_{n_d-2}$	$r_{n_d-3}$	$\dots$	$r_1$	$r_0$	
$r_0$	$r_1$	$r_2$	$\dots$	$r_{n_d-2}$	$r_{n_d-1}$	
$s_{n_d-2}$	$s_{n_d-3}$	$s_{n_d-4}$	$\dots$	$s_0$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$q_2$	$q_1$	$q_0$				

that the considered alternative description of the function  $\mathcal{T}_{wz}$  does not introduce any approximation, provided that the new parameter  $\tilde{\lambda}$  and the set  $\tilde{\Lambda}$  are properly constructed. This fact, which holds general validity, is illustrated by means of the following simple example.

*Example 1.* The LPV system given by the I/O representation:

$$\begin{aligned} z(t) + \lambda(t)z(t-1) &= 5\lambda^2(t)u(t-1) + 7\lambda^3(t)u(t-2), \\ \lambda(t) &\in \Lambda = [0, 5] \quad \forall t \in \mathbb{R}, \end{aligned}$$

can be equivalently written as

$$\begin{aligned} z(t) + \tilde{\lambda}_1(t)z(t-1) &= 5\tilde{\lambda}_2(t)u(t-1) + 7\tilde{\lambda}_3(t)u(t-2), \\ \tilde{\lambda}(t) \in \tilde{\Lambda} &= \left\{ \tilde{\lambda} \in \mathbb{R}^3 : \tilde{\lambda}_1 \in [0, 5], \tilde{\lambda}_2 = \tilde{\lambda}_1^2, \tilde{\lambda}_3 = \tilde{\lambda}_1^3 \right\} \quad \forall t \in \mathbb{R} \end{aligned}$$

Now, let us consider the following optimization problem

$$\tilde{\theta} = \arg \min_{\theta, P, \tilde{c}, \beta} \gamma \quad (19a)$$

s.t.

$$\begin{bmatrix} \tilde{c}^\top \tilde{d}(\tilde{\lambda}) + \tilde{d}^\top(\tilde{\lambda}) \tilde{c} - F(P) - \beta \tilde{c}^\top \tilde{c} & \tilde{v}^\top(\tilde{\lambda}) \\ \tilde{v}(\tilde{\lambda}) & \beta \gamma^2 \end{bmatrix} \succeq 0 \quad \forall \tilde{\lambda} \in \mathcal{P} \quad (19b)$$

$$\tilde{c} \in \mathcal{S}_{\tilde{c}} \quad (19c)$$

where

$$\tilde{d} = [\tilde{d}_{n_{\tilde{a}}}(\tilde{\lambda}) \quad \dots \quad \tilde{d}_2(\tilde{\lambda}) \quad \tilde{d}_1(\tilde{\lambda}) \quad 1] \quad (20)$$

$$\tilde{v} = [\tilde{v}_{n_{\tilde{v}}}(\tilde{\lambda}) \quad \dots \quad \tilde{v}_2(\tilde{\lambda}) \quad \tilde{v}_1(\tilde{\lambda}) \quad \tilde{v}_0(\tilde{\lambda})] \quad (21)$$

$$\tilde{c} = [\tilde{c}_{n_{\tilde{a}}} \quad \dots \quad \tilde{c}_2 \quad \tilde{c}_1 \quad 1] \quad (22)$$

and the set  $\mathcal{P} \supset \tilde{\Lambda}$  is a polytope of vertices  $\lambda_1^v, \lambda_2^v, \dots, \lambda_\rho^v$ .  $\mathcal{P}$  is a polytopic outer approximation of  $\tilde{\Lambda}$  derived by means of the procedure proposed in [8]. The problem considered in (19) leads to the following reformulation of Theorem 1.

**Theorem 2:** Given a polynomial  $\tilde{C}(q^{-1})$  of order  $n_{\tilde{a}}$

$$\tilde{C}(q^{-1}) = 1 + \tilde{c}_1 q^{-1} + \dots + \tilde{c}_{n_{\tilde{a}}} q^{-n_{\tilde{a}}}, \quad (23)$$

with roots inside the unite circle, the LPV system in Fig. 1 and with closed-loop polynomial (35) is quadratically stable in  $\mathcal{P}$  and satisfies the performance requirements in (5), if and only if there exists a symmetric matrix  $P \in \mathbb{R}^{n_{\tilde{a}} \times n_{\tilde{a}}}$  and a scalar  $\beta$  such that the following condition

$$\begin{aligned} \mathcal{J}(\theta, \lambda_i^v, P) &= \\ &= \begin{bmatrix} \tilde{c}^\top \tilde{d}(\lambda_i^v) + \tilde{d}^\top(\lambda_i^v) \tilde{c} - F(P) - \beta \tilde{c}^\top \tilde{c} & \tilde{v}^\top(\lambda_i^v) \\ \tilde{v}(\lambda_i^v) & \beta \gamma^2 \end{bmatrix} \succeq 0 \end{aligned} \quad (24)$$

is satisfied  $\forall i = 1, 2, \dots, \rho$ .

**Proof** We have to prove that condition (24) is a necessary and sufficient condition for guaranteeing that the LPV system in Fig. 1 is quadratically stable in  $\mathcal{P}$  and satisfies the performance requirements in (5) for all the trajectories of  $\tilde{\lambda}(t) \in \mathcal{P}$ . Necessity is obvious. In order to prove sufficiency, let us define the following function

$$\chi(w, \lambda) = w^\top \mathcal{J}(\theta, \lambda_i^v, P) w, \quad (25)$$

where  $w$  is any nonzero vector such that  $w \in \mathbb{R}^{n_d + n_v}$  and  $P$  is a symmetric matrix such that

$$\mathcal{J}(\theta, \lambda_i^v, P) \succeq 0 \quad \forall \tilde{\lambda}(t) \in \mathcal{P}. \quad (26)$$

As is well known, condition (26) is equivalent to

$$\chi(w, \lambda) \geq 0, \quad \forall \tilde{\lambda}(t) \in \mathcal{P}, \quad \forall w \in \mathbb{R}^{n_d + n_v}, \quad w \neq 0. \quad (27)$$

Due to the affine dependence on  $\tilde{\lambda}$ , the minimum of function  $\chi(w, \tilde{\lambda})$  over the polytope  $\mathcal{P}$  is attained at one of the vertexes  $\lambda_1^v, \lambda_2^v, \dots, \lambda_\rho^v$ , being  $\chi(w, \tilde{\lambda})$  a linear functional of  $\tilde{\lambda}$ . Proof of sufficiency follows from the fact that

$$\chi(w, \lambda_i^v) \geq 0 \quad \forall i = 1, \dots, \rho, \quad (28)$$

due to condition (24). ■

Thanks to Theorem (2), problem (19) simplifies to the finite dimensional nonconvex PMI problem

$$\tilde{\theta} = \arg \min_{\theta, P, \tilde{c}, \beta} \gamma \quad (29a)$$

s.t.

$$\begin{bmatrix} \tilde{c}^\top \tilde{d}(\lambda_i^v) + \tilde{d}^\top(\lambda_i^v) \tilde{c} - F(P) - \beta \tilde{c}^\top \tilde{c} & \tilde{v}^\top(\lambda_i^v) \\ \tilde{v}(\lambda_i^v) & \beta \gamma^2 \end{bmatrix} \succeq 0 \quad (29b)$$

$$\forall i = 1, \dots, \rho$$

$$\tilde{c} \in \mathcal{S}_{\tilde{c}} \quad (29c)$$

By exploiting a generalization of Descartes' rule ([14]), it is possible to apply the so-called *scalarization approach* proposed in [15], which allows us to replace the PMI constraints  $\mathcal{J}(\theta, \lambda_i^v, P) \succeq 0, \quad \forall i = 1, 2, \dots, \rho$  with a set of scalar multivariate polynomial constraints. Let us define the following matrix

$$\mathcal{H}_i(x) = \mathcal{J}(\theta, \lambda_i^v, P), \quad (30)$$

whose entries are scalar polynomials. The characteristic polynomial  $p_i(w, x)$  of  $\mathcal{H}_i(x)$  can be written, without loss of generality, as

$$\begin{aligned} p_i(w, x) &= \det(wI_{n_a+1} - \mathcal{H}_i(x)) = \\ &= w^{n_a+1} + \sum_{k=1}^{n_a+1} (-1)^k h_k(x) w^{n_a+1-k}, \end{aligned} \quad (31)$$

for all  $w \in \mathbb{R}$ , where  $h_k(x)$  are scalar polynomials and  $I_{n_a+1}$  denotes the identity matrix of size  $n_a+1$ . Application of the *scalarization approach* to matrix  $\mathcal{H}_i(x)$  leads to the following result.

**Result 2: Scalarization of PMI [15]**

$$\mathcal{H}_i(x) \succeq 0 \text{ iff } h_k(x) \geq 0 \quad \forall k = 1, \dots, n_a+1. \quad \blacksquare$$

By Result 2, problem (29) is converted into a standard polynomial optimization problem which can be solved by exploiting the relaxation-based approaches proposed in [16] and [17]. An alternative approach to solve problem (29) is based on the LMI relaxations for PMI problems proposed

in [18] in the spirit of the results of sum-of-squares decompositions of positive polynomial matrices discussed in [19] and [20]. In particular, in [18], PMI constraints of the kind  $\mathcal{H}_i(x) \succeq 0$  are directly handled in matrix form without conversion to a set of scalar polynomials and a sequence of convex-relaxed problems is built up, whose solutions are guaranteed to converge to the global optimum of the original nonconvex PMI problem. Such an approach allows one to detect if the global optimum of the original PMI problem (29) is reached, and if so, to extract global minimizers. From a computational point of view, this methodology is in general more convenient than the *scalar approach* (see [18] for a detailed discussion). Such an approach can be efficiently implemented in Matlab by using the Gloptipoly toolbox [21].

**Remark 1:** It is worth noting that computational complexity of problem (29) can be significantly reduced if, instead of minimizing  $\gamma$ , we choose to look for a controller satisfying the constraints of problem (29) for a given constant value of  $\gamma$ , defined by the user on the basis of the desired  $L_2$ -induced gain. In such a case, the general PMI optimization problem becomes a simpler BMI feasibility problem. Minimization of  $\gamma$  can be performed by means of a bisection-like procedure.

## V. A SIMULATION EXAMPLE

In this section we aim at demonstrating the effectiveness of the presented approach through a simple example. The system to be controlled is a discrete-time second-order LPV system  $\mathcal{G}(q^{-1}, \lambda)$  described by:

$$\mathcal{G}(q^{-1}, \lambda) : y_t = -a_1(\lambda_t)y_{t-1} - a_2(\lambda_t)y_{t-2} + b_0(\lambda_t)u_t + b_1(\lambda_t)u_{t-1} + b_2(\lambda_t)u_{t-2},$$

where

$$a_1(\lambda_t) = -0.2 + 0.7\lambda_t, \quad a_2(\lambda_t) = 0.7 + 0.4\lambda_t,$$

$$b_0(\lambda_t) = 0.6 + 0.2\lambda_t, \quad b_1(\lambda_t) = 3.4 + 1.2\lambda_t, \quad b_2(\lambda_t) = 1.6 + 2.8\lambda_t.$$

The scheduling parameter  $\lambda$  is assumed to take values in the range  $[0, 1]$ . Note that the system to be controlled is not stable. For instance, when  $\lambda_t = 1$  for all  $t = 1, 2, \dots$ , the corresponding frozen LTI system is unstable. A discrete-time LPV PI controller  $\mathcal{K}(q^{-1}, \lambda)$  described by the difference equation:

$$\mathcal{K}(q^{-1}, \lambda) : (1 - q^{-1})u(t) = [(1 - q^{-1})(K_{p0} + K_{p1}\lambda_t) + (K_{i0} + K_{i1}\lambda_t)q^{(-1)}]e(t), \quad (33)$$

is then designed in order to guarantee quadratic stability of the closed-loop system, and also that the  $L_2$ -gain from the reference signal  $r$  to the output signal  $y$  is smaller than or equal to  $\gamma = 1.6$  for all possible trajectories of the scheduling variable  $\lambda$  in the interval  $\Lambda = [0, 1]$ . In (33),  $K_p(\lambda_t)$  is the proportional part of the controller and  $K_i(\lambda_t)$  is the weight on the integral action, whose descriptions are given, respectively, by

$$K_p(\lambda_t) = K_{p0} + K_{p1}\lambda_t, \quad K_i(\lambda_t) = K_{i0} + K_{i1}\lambda_t. \quad (34)$$

Such a structure of the controller leads to a closed-loop mapping  $\mathcal{T}_{yr}(q^{-1}, \lambda, t)$ , between the reference signal  $r$  and the

output signal  $y$ , that dynamically depends on the scheduling parameter  $\lambda$ . However, in this example, in order to apply the approach discussed in the paper, the following closed-loop mapping  $\mathcal{T}_{yz}(q^{-1}, \lambda)$  with static dependance on  $\lambda$  is considered in designing the controller:

$$\mathcal{T}_{yr}(q^{-1}, \lambda) : \mathcal{D}(q^{-1}, \lambda(t))y(t) = \mathcal{V}(q^{-1}, \lambda(t))r(t), \quad (35)$$

with

$$\begin{aligned} \mathcal{D}(q^{-1}, \lambda(t)) &= 1 + b_0(\lambda_t)K_p(\lambda_t) + \\ &[b_1(\lambda_t)K_p(\lambda_t) + b_0(\lambda_t)K_i(\lambda_t) - b_0(\lambda_t)K_p(\lambda_t) + a_1(\lambda_t) - 1]q^{-1} + \\ &[b_2(\lambda_t)K_p(\lambda_t) + b_1(\lambda_t)K_i(\lambda_t) - b_1(\lambda_t)K_p(\lambda_t) + a_2(\lambda_t) - a_1(\lambda_t)]q^{-2} + \\ &[b_2(\lambda_t)K_i(\lambda_t) - b_2(\lambda_t)K_p(\lambda_t) - a_2(\lambda_t)]q^{-3}, \end{aligned} \quad (36)$$

and

$$\begin{aligned} \mathcal{V}(q^{-1}, \lambda(t)) &= b_0(\lambda_t)K_p(\lambda_t) + \\ &[b_1(\lambda_t)K_p(\lambda_t) + b_0(\lambda_t)K_i(\lambda_t) - b_0(\lambda_t)K_p(\lambda_t)]q^{-1} + \\ &[b_2(\lambda_t)K_p(\lambda_t) + b_1(\lambda_t)K_i(\lambda_t) - b_1(\lambda_t)K_p(\lambda_t)]q^{-2} + \\ &[b_2(\lambda_t)K_i(\lambda_t) - b_2(\lambda_t)K_p(\lambda_t)]q^{-3}. \end{aligned} \quad (37)$$

Such a closed-loop mapping can be proved to be a good approximation of the true closed-loop mapping  $\mathcal{T}_{yr}(q^{-1}, \lambda, t)$  for slow variations of the scheduling parameter  $\lambda$ . It is worth remarking that  $\mathcal{D}(q^{-1}, \lambda(t))$  and  $\mathcal{V}(q^{-1}, \lambda(t))$  in (36) and (37), respectively, quadratically depend on  $\lambda$ . As discussed in Section IV, the new augmented scheduling variable  $\tilde{\lambda} = [\tilde{\lambda}_1, \tilde{\lambda}_2]^T = [\lambda, \lambda^2]^T \in \mathbb{R}^2$  can be defined so that the relation between the reference signal  $r$  and the output  $y$  is an LPV model with affine dependence on  $\tilde{\lambda}$ . Indeed, the augmented scheduling parameter vector  $\tilde{\lambda}$  lies in the set  $\tilde{\Lambda} \subseteq \mathbb{R}^2$  that is the arc of the parabola  $\tilde{\lambda}_2 = \tilde{\lambda}_1^2$  confined to the interval  $\tilde{\lambda}_1 \in [0, 1]$ , i.e.  $\tilde{\Lambda} = \{(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \mathbb{R}^2 : 0 \leq \tilde{\lambda}_1 \leq 1; \tilde{\lambda}_2 = \tilde{\lambda}_1^2\}$ . A polytopic outer approximation  $\mathcal{P}$  of  $\tilde{\Lambda}$  is then sought by means of the procedure proposed in [8]. The set  $\tilde{\Lambda}$  and its polytopic outer approximation  $\mathcal{P}$  are plotted in Fig. 2. For every vertex of the polytope  $\mathcal{P}$ , the LMI constraints in problem (29) are enforced and the following parameters are obtained:  $K_p(\lambda_t) = K_{p0} + K_{p1}\lambda_t = 1.9687 - 0.9109\lambda_t$ ,  $K_i(\lambda_t) = K_{i0} + K_{i1}\lambda_t = 0.2640 + 0.5652\lambda_t$ . The designed controller is tested via simulation for the trajectory of the scheduling parameter  $\lambda$  reported in Fig. 3(d). The evolution of the output signal  $y$  and of the tracking error  $e$  of the LPV closed-loop system are plotted in Figs. 3(a) and 3(b), respectively, which show a satisfactory tracking performance for different operating conditions. In particular,

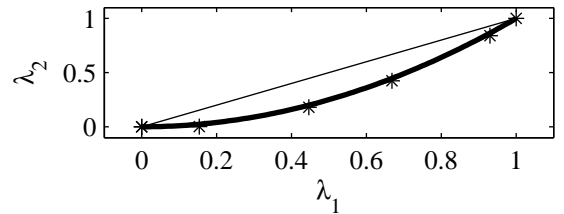


Fig. 2. Nonconvex semialgebraic set  $\tilde{\Lambda}$  (arc of parabola, thick line) and polytopic outer approximation  $\mathcal{P}$  (region inside thin line). Vertices of  $\mathcal{P}$  are marked with stars (\*).

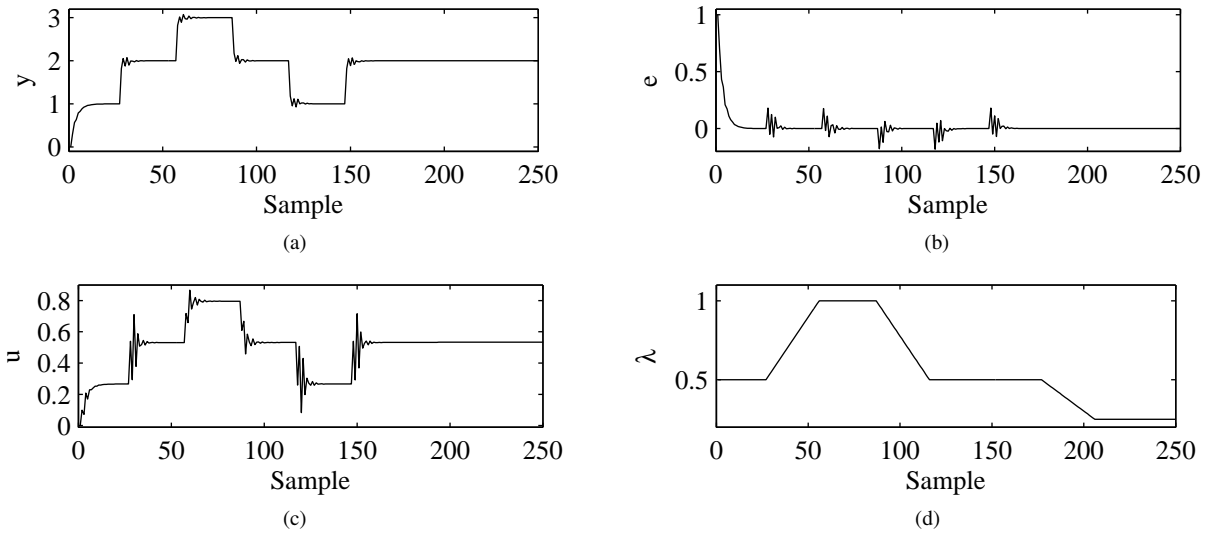


Fig. 3. (a) Closed-loop output signal  $y$  (solid line); (b) tracking error  $e$ ; (c) input signal  $u$ ; (d) scheduling parameter  $\lambda$ .

the percentage overshoot is smaller than 12% and the time required by the output to stay within a relative error range of  $[-0.01, 0.01]$  with respect to its steady-state value is less than 15 time samples. Besides, the integral action of the controller guarantees zero steady-state tracking error. The control input  $u$  is plotted in Fig. 3(c). It is worth remarking that, as commonly done in practice, a rough outer approximation of the semialgebraic set  $\tilde{\Lambda}$  given by the box  $B = \{(\tilde{\lambda}_1, \tilde{\lambda}_2) \in \mathbb{R}^2 : 0 \leq \tilde{\lambda}_1 \leq 1, 0 \leq \tilde{\lambda}_2 \leq 1\}$  could be alternatively considered in computing the controller  $\mathcal{K}(q^{-1}, \lambda)$ . Unfortunately, because of the conservativeness introduced by the box  $B$ , it is not possible, in this case, to compute a quadratic stabilizer controller  $\mathcal{K}(q^{-1}, \lambda)$  such that the  $L_2$ -gain from the reference signal  $r$  to the closed-loop output signal  $y$  is smaller than or equal to  $\gamma = 1.6$ . In other words, evaluation of the LMI constraints (19b) at the vertexes of the box  $B$  leads to an infeasible problem for  $\gamma = 1.6$ .

## VI. CONCLUSIONS

A new controller synthesis approach for linear parameter varying systems given in input-output representation is presented in the paper. The proposed procedure can be applied under the quite general assumption that the closed-loop system depends polynomially on the scheduling variables. By introducing a new augmented scheduling parameter and by applying a suitable procedure for polytopic outer approximations of semialgebraic sets, the synthesis problem is reformulated as a finite dimension polynomial-matrix-inequality problem which is solved by means of convex relaxation techniques.

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